Best linear forecast of volatility in financial time series

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(Received 20 January 2004; published 3 September 2004)

The autocorrelation function of volatility in financial time series is fitted well by a superposition of several exponents. This case admits an explicit analytical solution of the problem of constructing the best linear forecast of a stationary stochastic process. We describe and apply the proposed analytical method for forecasting volatility. The leverage effect and volatility clustering are taken into account. Parameters of the predictor function are determined numerically for the Dow Jones 30 Industrial Average. Connection of the proposed method to the popular autoregressive conditional heteroskedasticity models is discussed.

DOI: 10.1103/PhysRevE.70.036102 PACS number(s): 89.65.Gh, 89.75.Da, 02.50.Ng, 02.50.Sk

I. INTRODUCTION

The methods developed in studying complex physical systems have been successfully applied for decades to analyze financial data [1–3]. The quantitative study of financial data continues to attract growing interest motivated by the existence of universal features in the dynamics of different markets, such as power-law tails of the return distributions [4–11], scaling as a first approximation [3], deviations from scaling of the empirical return distributions [5,8,12], volatility clustering [12,13], and the leverage effect [14–16,24]. Phenomenological and microscopic models [8–11,19–22] have been proposed to explain the established conventional facts. The field of research connected to modeling financial markets has been named econophysics.

A stock's volatility represents the simplest measure of its riskiness or uncertainty. Formally, the volatility is the annualized standard deviation of the stock's returns during the period of interest. The random walk model proposed by Bachelier in 1900 [1] presupposes a constant volatility. There is ample empirical evidence, however, that the volatility is not a constant, but represents a random variable. Two well established "stylized" facts concerning the volatility are long ranged volatility-volatility correlations, which are also known as volatility clustering [13], and return-volatility correlations, which are also known as the leverage effect [14,15].

The volatility is a key variable for controlling risk measures associated with the dynamics of prices of financial assets. The implied volatility extracted from options prices represents a market estimate of future volatility. Pure exposure to future volatility is provided by volatility swaps [17,18]. The volatility enters all options pricing models, so its knowledge has great value for estimating the price distributions of the equilibrium options state.

Volatility clustering manifests itself in the occurrence of large changes of the index at neighboring times (observed localized outbursts). The leverage effect has its origin in the observed negative correlation between past returns and future volatility. A possible explanation of this effect [14–16] is due to the fact that negative returns increase financial leverage and extend the risk for investors and thereby a stock's volatility. A statistical study [23,24] demonstrated clearly that the leverage effect is one directional: past returns correlate with future volatility only.

In this paper, we propose an analytical method to evaluate future volatility as a linear function of the lagged volatility and lagged returns. The method takes volatility clustering and the leverage effect into account and provides for stationary stochastic processes the smallest forecasting error in the class of all linear functions. In this precise sense, we talk about the best linear forecast (BLF) of the volatility.

The BLF problem for a stationary stochastic process was formulated and solved by Kolmogorov [25] in 1941 and by Wiener [26] in 1949. A modern review of BLF methods can be found in Ref. [27]. We apply these methods to construct the BLF volatility function for the Dow Jones 30 Industrial Average (DJIA).

The outline of the paper is as follows. In the next section, we remove the leverage effect from the original time series to work with a reduced volatility $\chi(t)$ that has by definition a vanishing covariance with past returns. The spectral density of a stochastic process can be factorized, $f(\omega)=|\varphi(\omega)|^2$, if its correlation function represents a superposition of exponential functions. An explicit expression is derived for the amplitude $\varphi(\omega)$. The analytical properties of the amplitude $\varphi(\omega)$ in the complex ω plane are important to provide an explicit representation of the predictor function. In Sec. III, the BLF problem is analyzed further to account for reduced volatility clustering and to construct the BLF function. In Sec. IV, we fit more than 100 years of data of the daily historical volatility of the DJIA in order to determine the parameters of the BLF function. Numerical estimates are given to illustrate the method developed. The minimization of the forecasting error for the reduced volatility predictor function is shown to be equivalent to the minimization of the forecasting error of the original volatility time series. An explicit expression for the forecasting error is given. In the Conclusion, a connection of the BLF method with the autoregressive conditinal heteroskedasticity (ARCH) models [28–33], in which future variance is also represented as a linear combination of past observables, is discussed.

II. FACTORIZATION OF SPECTRAL DENSITY

The evolution of a market index value or a stock price $S(t)$ is described by the equation (see, e.g., [34])

$$\frac{dS(t)}{S(t)} = \mu dt + d\psi(t). \tag{1}$$

The value $d\psi(t)$ is a noise added to the path followed by $S(t)$, with the expectation value $E[d\psi(t)]=0$ and the variance $\mathrm{var}[d\psi(t)]=\sigma(t)^2 dt$. The volatility $\sigma(t)$ represents a generic measure of the magnitude of market fluctuations. We consider a discrete version of the random walk problem by setting $dt=1$, $d\psi(t)=\xi(t)$, and $dS(t)=S(t)-S(t-1)$. The sampling intervals are enumerated by the integer time parameter t.

The volatility $\sigma(t)$ is a hidden variable and its extraction form the market observables is a separate difficult task. A possible estimator $\eta(t)=|\xi(t)|$ of the volatility is defined in terms of returns,

$$\xi(t) = (S(t) - S(t-1))S(t)^{-1} - \mu. \tag{2}$$

In what follows, the term "volatility" refers to the estimator $\eta(t)=|\xi(t)|$; the annualizing factor will not apply. Use of the variance estimator $v(t)=|\xi(t)|^2$ would complexify the problem due to divergences connected to the existence of power-law tails (the "variance of a variance" is infinite, $\mathrm{var}[\xi^2]=\infty$, since $dF(\xi)\sim d\xi/\xi^4$ at $\xi \gg 1$; see, e.g., [7]). At large time scales, different estimators are expected to be close to volatility $\sigma(t)$ and to each other. The problem of the efficiency of the volatility estimators is postponed for other studies.

It is usually assumed that financial time series constitute stationary stochastic processes, the autocorrelation functions of which depend on the relative time only. The stock evolution problem is assumed therefore to be invariant with respect to time translations.

First, we remove from the time series $\eta(t)$ the leverage effect using the variable $\chi(t)$:

$$\chi(t) = \eta(t) - \sum_s \mathrm{cov}[\eta(0), \xi(-s)]\mathrm{var}^{-1}[\xi]\xi(t-s). \tag{3}$$

The decomposition (3) has predictive power, since $\mathrm{cov}[\eta(t), \xi(t-s)] \sim \theta(s)$, so $\chi(t)$ depends on the lagged price increments only. Note that $E[\eta]=E[\chi]$, since $E[\xi]=0$. Due to the definition (3) and by virtue of the equation

$$\mathrm{cov}[\xi(t), \xi(s)] = \delta_{ts}\mathrm{var}[\xi], \tag{4}$$

which holds true for sampling intervals greater than 20 min [7,12], we have

$$\mathrm{cov}[\chi(t), \xi(s)] = 0. \tag{5}$$

The reduced volatility $\chi(t)$ does not experience the leverage effect. So its predictor depends on the past $\chi(t)$ only. It is possible therefore to focus on volatility clustering only, while the leverage effect is taken into account explicitly through Eq. (3).

The autocorrelation function $B(t)=\mathrm{cov}[\chi(t), \chi(0)]/E[\chi^2]$ of a stationary stochastic process $\chi(t)$ can be fitted in many cases by a superposition of exponents

$$B(t) = \sum_{i=1}^{n} d_i e^{-\alpha_i |t|}. \tag{6}$$

The best linear forecast of the observable $\chi(t)$ in such a case simplifies substantially. The case $n=1$ is discussed in Ref. [27]. We provide a solution of the BLF problem for arbitrary values of n.

The spectral density $f(\omega)$ of the stochastic process $\chi(t)$ is given by the Fourier transform of the correlation coefficient (6):

$$f(\omega) = \sum_{t=-\infty}^{+\infty} e^{-i\omega t} B(t) = \sum_{i=1}^{n} d_i \frac{1 - e^{-2\alpha_i}}{(1 - e^{-\alpha_i} u)(1 - e^{-\alpha_i}/u)}, \tag{7}$$

where $u=\exp(-i\omega)$. The function $f(\omega)$ can be represented in the form

$$f(\omega) = P_{n-1}(\phi)\left[\prod_{i=1}^{n}(1 - e^{-\alpha_i} u)\left(1 - e^{-\alpha_i}\frac{1}{u}\right)\right]^{-1}, \tag{8}$$

where $\phi=(1/2)(u+1/u)$ and

$$P_{n-1}(\phi) = 2^n \exp\left(-\sum_{i=1}^{n}\alpha_i\right)\sum_{i=1}^{n} d_i \sinh(\alpha_i)\prod_{k\neq i}^{n}[\cosh(\alpha_i) - \phi]$$

$$= D_n 2^{n-1} \exp\left(-\sum_{i=1}^{n-1}\nu_i\right)\prod_{i=1}^{n-1}[\cosh(\nu_i) - \phi]. \tag{9}$$

The amplitude $\varphi(u)$ such that $f(\omega)=\varphi(u)\varphi(u)^*$ can be chosen to be analytical, rational, and regular at $|u|<1$:

$$\varphi(u) = D_n^{1/2}\prod_{i=1}^{n-1}(1 - e^{-\nu_i} u)\left(\prod_{i=1}^{n}(1 - e^{-\alpha_i} u)\right)^{-1}. \tag{10}$$

The additive representation

$$\varphi(u) = \sum_{i=1}^{n} c_i \frac{1}{1 - \beta_i u} \tag{11}$$

is completely equivalent to the multiplicative representation (10). Here, $\beta_i=e^{-\alpha_i}$ and

$$D_n = 2\exp\left(-\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n-1}\nu_i\right)\sum_{i=1}^{n} d_i \sinh(\alpha_i),$$

$$c_i = D_n^{1/2}\prod_{k=1}^{n-1}(e^{-\alpha_i} - e^{-\nu_k})\left(\prod_{k\neq i}^{n}(e^{-\alpha_i} - e^{-\alpha_k})\right)^{-1},$$

$$\cosh(\nu_i) = \phi_0^i, \tag{12}$$

where ϕ_0^i are $n-1$ roots of one equation $P_{n-1}(\phi_0^i)=0$. For $n=1$, $c_1=d_1^{1/2}\sqrt{1-\beta_1^2}$. Analytical solutions for ν_i exist up to $n=5$.

Knowledge of the Fourier transform of the autocorrelation function is not sufficient for a complete reconstruction of the Fourier transform of the stochastic process $\chi(t)$. If the autocorrelation function represents a superposition of the exponents (6), the problem admits a solution $\varphi(u)$, such that

$f(u)=|\varphi(u)|^2$, in the class of rational functions. If we require further that the function $\varphi(u)$ be regular at $|u|\leqslant 1$, an unambiguous solution $\varphi(u)$ can be provided. This solution coincides with the Fourier transform of the time series $\chi(t)$ up to a phase factor. It is remarkable that we need not know the phase, since all the relevant information is contained in the spectral density $f(\omega)$. The BLF problem then simplifies considerably due to the special analytical properties of the function $\varphi(u)$.

III. BLF FUNCTION

The correlation function corresponding to the spectral density (7) can be found from the inverse Fourier transform. In the case of Eq. (10), we consider first $t>0$:

$$B(t)=\int_{-\pi}^{\pi} e^{i\omega t}\varphi(\omega)\varphi(-\omega)\frac{d\omega}{2\pi}=-\int_{\mathcal{C}_r}\frac{1}{u^{t+1}}\varphi(u)\varphi\left(\frac{1}{u}\right)\frac{du}{2\pi i}, \tag{13}$$

where $\mathcal{C}_r=\{e^{-i\omega},\omega=-\pi\cdots\pi\}$. The poles of $\varphi(u)$ are located at $|u|\geqslant R=\min\{e^{\alpha_i}\}$, while the poles of $\varphi(1/u)$ are located at $|u|\leqslant 1/R$. We move the contour $\mathcal{C}_r$ to infinity and get

$$B(t)=\sum_{i=1}^{n}\sum_{k=1}^{n}\beta_i^{|t|}\frac{c_i c_k}{1-\beta_i\beta_k}. \tag{14}$$

Comparison with Eq. (6) gives

$$d_i=\sum_{k=1}^{n}\frac{c_i c_k}{1-\beta_i\beta_k}. \tag{15}$$

The same result (14) comes out at $t<0$. For $n=1$, $B(t)=\beta_1^{|t|}c_1^2/(1-\beta_1^2)$.

A stochastic process $\chi(t)$ can be represented as a linear combination of a normally distributed uncorrelated sequence $\zeta(t)\sim N(0,\sigma_\chi)$ with $\sigma_\chi^2=E[\chi^2]$,

$$\chi(t)=E[\chi]+\sum_{s=0}^{+\infty}C(s)\zeta(t-s), \tag{16}$$

provided that the spectral function admits the factorization and the amplitude $\varphi(u)$ is regular at $|u|=1<R$ (see, e.g., [27]). The expansion coefficients equal

$$C(t)=\sum_{i=1}^{n}c_i\beta_i^t. \tag{17}$$

It is remarkable that only retarded $\zeta(s)$ enter the summation in Eq. (16). This is a consequence of the convergence of the Taylor expansion of the amplitude $\varphi(u)$ at $|u|=1$, which is in turn a consequence of the analyticity at $|u|<R$: The convergence radius of the expansion is associated with the first pole at $|u_1|=R$. The stationary stochastic process $\chi(t)$ can be interpreted as a result of filtering the normal sequence $\zeta(t)$.

The BLF function for the time horizon τ has the form [27]

$$\hat{\chi}_\tau(t)=E[\chi]+\sum_{s=\tau}^{+\infty}\frac{\Xi_\tau(0)^{(s)}}{s!}\{\chi(t-s)-E[\chi]\}. \tag{18}$$

The weight coefficients $\Xi_\tau(0)^{(s)}$ are derivatives of the function $\Xi_\tau(u)=\varphi_\tau(u)/\varphi(u)$ at $u=0$. Here,

$$\varphi_\tau(u)=\sum_{s=\tau}^{+\infty}C(t)u^s=\sum_{i=1}^{n}c_i(\beta_i u)^\tau\frac{1}{1-\beta_i u}.$$

For constructing a linear prognosis function, the overall normalization factor in $B(t)$ is not important, since it drops out from the ratio $\varphi_\tau(u)/\varphi(u)$. In virtue of Eq. (5),

$$\mathrm{cov}[\hat{\chi}_\tau(t),\xi(s)]=0. \tag{19}$$

In terms of the stochastic process $\zeta(t)$, the BLF function looks like

$$\hat{\chi}_\tau(t)=E[\chi]+\sum_{s=\tau}^{+\infty}C(s)\zeta(t-s). \tag{20}$$

At $s=\tau$ we obtain

$$\frac{\Xi_\tau(0)^{(\tau)}}{\tau!}=\frac{\sum_{i=1}^{n}c_i e^{-\alpha_i\tau}}{\sum_{i=1}^{n}c_i}, \tag{21}$$

and at $l=s-\tau>0$

$$\frac{\Xi_\tau(0)^{(s)}}{s!}=D_n^{-1/2}\sum_{j=1}^{n-1}e^{-\nu_j(l-1)}\left(\prod_{i\neq j}^{n-1}(e^{-\nu_j}-e^{-\nu_i})\right)^{-1}\times\sum_{i=1}^{n}c_i e^{-\alpha_i\tau}\prod_{k\neq i}^{n}(e^{-\nu_j}-e^{-\alpha_k}). \tag{22}$$

The last two equations complete the solution of the BLF problem for the case when the correlation function is a superposition of the exponent functions (6). The $n(n-1)$ terms in the right side of Eq. (22) are not all positive definite.

IV. PARAMETERS OF BLF VOLATILITY FUNCTION FOR DOW JONES 30 INDUSTRIAL AVERAGE

Let us apply the BLF method to forecasting the volatility for the DJIA. The daily returns are defined by Eq. (2) where $S(t)$ are the DJIA index close values, the volatility equals $\eta(t)=|\xi(t)|$, and $\chi(t)$ is defined by Eq. (3). In Fig. 1, we show the empirical values of the correlation coefficients $\mathrm{corr}[\eta(t),\eta(0)]$, $\mathrm{corr}[\eta(t),\xi(0)]$ and $\mathrm{corr}[\chi(t),\chi(0)]$, $\mathrm{corr}[\chi(t),\xi(0)]$ and the exponential fit of the correlation coefficient $\mathrm{corr}[\chi(t),\chi(0)]$ versus the time lag t. Let us recall that $\mathrm{corr}[A,B]=\mathrm{var}[A,B]/\sqrt{E[A^2]E[B^2]}$ and $-1\leqslant\mathrm{corr}[A,B]\leqslant 1$. To calculate $\chi(t)$, we run the summation over s in Eq. (3) from 0 to 250 and use the empirical correlation function of $\eta(t)$ and $\xi(s)$ without additional smearing. Up to $t=250$, the correlation coefficient $\mathrm{corr}[\chi(t),\xi(0)]$ is less noisy as compared with other correlators. The parameters d_i and α_i

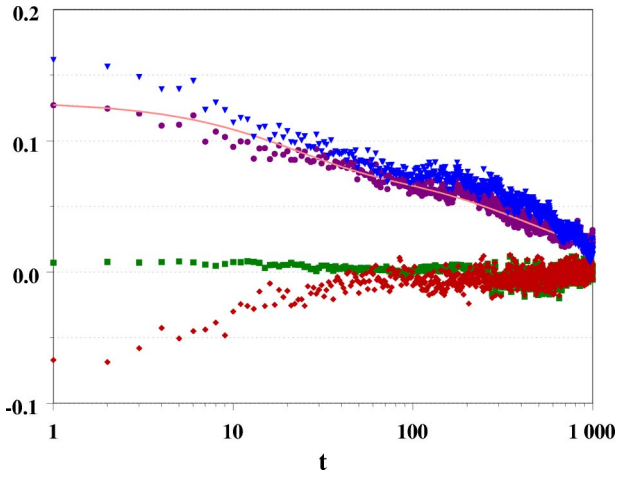

FIG. 1. Empirical correlation coefficients corr$[\eta(t),\eta(0)]$, corr$[\eta(t),\xi(0)]$ and corr$[\chi(t),\chi(0)]$, corr$[\chi(t),\xi(0)]$ versus the number of trading days t of the Dow Jones 30 Industrial Average. The reduced volatility $\chi(t)$ is defined through Eq. (3). The correlation coefficients are calculated using more than 100 years of the daily quotes, starting on 26 May 1896 and ending on 31 December 1999 (i.e., a total of 28 507 trading days). The values of corr$[\eta(t),\eta(0)]$ and corr$[\eta(t),\xi(0)]$ are denoted, respectively, by triangles and diamonds. The values of corr$[\chi(t),\chi(0)]$ and corr$[\chi(t),\xi(0)]$ are denoted by circles and boxes. The solid curve is the exponential fit (6) with parameters given in Table I. It is seen that corr$[\chi(t),\xi(0)]\approx 0$. The leverage effect is thus removed from $\chi(t)$.

are listed in Table I. The equation $P_3(\phi)=0$ determines the parameters ν_i $(i=1,2,3)$ according to Eq. (9). Using Eqs. (12), we find the values of D_4 and c_i, which we also show in Table I. One can check that Eq. (15) is satisfied. The correlation coefficient corr$[\chi(t),\chi(0)]$ drops from $0.53(=\Sigma_{i=1}^{4}d_i)$ to $0.13(=\Sigma_{i=2}^{4}d_i)$ when t changes by one unit from $t=0$ to $t=1$. The value of α_1 is therefore large and can be fixed by considering high-frequency data only. The results shown on Fig. 1 and in Table I and II are obtained for $\alpha_1=+\infty$. The values of $1/\alpha_i$ for $i=2,3,4$ are equal to about one month, one year, and four calendar years, respectively.

The average fitting error 13.6% can be compared to the average noise of 11.1% in the DJIA index. The latter is estimated as the variance of $\chi(t)-[\chi(t-1)+\chi(t+1)]/2$. The quality of fit presented in Fig. 1 is therefore reasonable.

The weight coefficients $\Xi_\tau(0)^{(s)}$ can be found with the use of Eq. (21) at $s=\tau$ and Eq. (22) at $s>\tau$. We show values of

TABLE I. Parameters d_i and α_i entering the fit of the autocorrelation function (6) of the reduced volatility, parameters ν_i which determine the roots of equation $P_{n-1}(\phi_0)=0$, and parameters c_i which determine the additive representation (11) of the function $\varphi(u)$. The value of D_4 represents the normalization constant according to Eq. (12).

d_1	0.40	α_1	$+\infty$	ν_1	0.002257	c_1	0.606241
d_2	0.05	α_2	1/20	ν_2	0.012107	c_2	0.038233
d_3	0.03	α_3	1/250	ν_3	0.125302	c_3	0.007857
d_4	0.05	α_4	1/1000	D_4	0.435341	c_4	0.007473

TABLE II. The weight coefficients (21) and (22) of the BLF function for some values of the parameters l and τ. Here, τ is the forecast horizon and $l=s-\tau$ is the number of trading days entering the predictor function starting from the most recent day.

l	$\tau=1$	$\tau=2$	$\tau=10$	$\tau=100$
0	0.07830	0.07555	0.05780	0.01862
1	0.06942	0.06702	0.05150	0.01710
2	0.06158	0.05949	0.04594	0.01575
10	0.02432	0.02366	0.01941	0.00916

the weight coefficients divided by $s!$ in Table II for $l=s-\tau=0$, 1, 2, and 10 and $\tau=1$, 2, 10, and 100.

The BLF volatility function looks like

$$\hat{\eta}_\tau(t)=E[\chi]+\sum_{s=\tau}^{+\infty}\frac{\Xi_\tau(0)^{(s)}}{s!}\{\chi(t-s)-E[\chi]\}+\sum_{s=\tau}^{+\infty}\mathrm{cov}[\eta(0),\xi(-s)]\mathrm{var}^{-1}[\xi]\xi(t-s) \quad (23)$$

where the unknown future returns are set equal to zero: $\xi(t-s)\to E[\xi(t-s)]=0$ for $0\leqslant s<\tau$.

Using Eqs. (4), (5), and (19), one gets

$$E[\{\hat{\eta}_\tau(t)-\eta(t)\}^2]=E[\{\hat{\chi}_\tau(t)-\chi(t)\}^2]+\sum_{s=0}^{\tau-1}\mathrm{cov}[\eta(0),\xi(-s)]^2\mathrm{var}^{-1}[\xi], \quad (24)$$

so the minimization of the $\hat{\chi}_\tau(t)$ error according to Eq. (18) is equivalent to the minimization of the $\hat{\eta}_\tau(t)$ error. Using decompositions (16) and (20), the $\hat{\chi}_\tau(t)$ error can be evaluated as

$$E[\{\hat{\chi}_\tau(t)-\chi(t)\}^2]=E[\chi^2]\sum_{s=0}^{\tau-1}C^2(s)=E[\chi^2]\sum_{i=1}^{n}\sum_{k=1}^{n}c_ic_k\frac{1-e^{-(\alpha_i+\alpha_k)\tau}}{1-e^{-(\alpha_i+\alpha_k)}}. \quad (25)$$

[There is a misprint in Eq. (10.2) of Ref. [27].] At $\tau\to+\infty$, $E[\{\hat{\chi}_\tau(t)-\chi(t)\}^2]\to E[\chi^2]\Sigma_{i=1}^{n}d_i=\mathrm{var}[\chi]$, in agreement with the fact that $\hat{\chi}_\tau(t)\to E[\chi]$. Arguments of this kind do not apply to the variance estimator $v(t)=|\xi(t)|^2$, since var$[v]=\infty$ due to the power-law tails of the return distributions [7].

Nonlinear models for volatility forecasting [19,20], which take into account, in addition to the volatility clustering and leverage effect, also the heavy tails of the return distributions and the approximate scaling, represent an alternative class of stochastic volatility models. The efficiency of such models can be tested in general using Monte Carlo simulations and/or back tests over historical data. The approach of Refs. [19,20] is more general, since it allows a calculation of the probability density function of the volatility. The BLF method predicts the average volatility only. It can, however,

be extended to forecasting $|\xi|^a$ for arbitrary $0<a$ such that $E[|\xi|^{2a}]<\infty$. If all moments $E[|\xi|^a]$ of the future distribution are known, the reconstruction of the probability density function of the volatility must be possible within the BLF method also.

The high-frequency data have a pronounced intraday structure. At the opening, European markets have a huge volatility, close to the opening of the USA markets and slightly prior to it. The volatility correlators have oscillating components at an intraday scale. These oscillations are seen also in the Fourier spectrum of the volatility, with several pronounced harmonics. The proposed method cannot be applied without modifications to the intraday data, since oscillations cannot be fitted by a superposition of exponential functions.

The hypothesis of a stationary character of the DJIA index volatility, if tested rigorously against all alternatives, should take into account the long-ranged correlations, as a result of which the elements of financial time series cannot be treated as independent. If correlations are nonetheless neglected, the estimates of the DJIA volatility over different few-year intervals give scattered results with variations significantly above the statistically allowed level for a Gaussian random walk. We have, however, not observed using this simplified method any obvious trend for the DJIA volatility over the last more than 100 years.

V. CONCLUSION

The BLF problem for a stationary stochastic process was formulated in 1941 by Kolmogorov [25] and later by Wiener [26]. A modern review of BLF methods can be found in Ref. [27]. In this paper, we reported an explicit analytical solution of the BLF problem for the practically important case when the autocorrelation function represents a superposition of exponential functions. The autocorrelation function of the volatility in a financial time series is known to be fitted well by such a superposition. We applied the results obtained to construct the BLF volatility function for the DJIA.

The popular autoregressive conditional heteroskedasticity models of time dependent volatility, proposed by Engle [28] (for a review, see [29–33]), describe the variance $\sigma(t)^2$ as a linear function of the past observables. The ARCH models are conceptually very close to the BLF approach. Equation (23) expresses the forecasting volatility also as a linear function of the past volatility and past returns. Equation (23) gives, however, the best linear forecast with the proved smallest forecasting error (25). The weight coefficients $\Xi_\tau(0)^{(s)}$ allow one to evaluate the magnitude and number of terms needed for the ARCH models to quantify future variance with sufficiently good precision. The ARCH models receive additional support and a more general framework through the BLF formula (23).

Accurate estimates of the future volatility are important for risk management and options pricing. The BLF formula (23) is interesting as the proved most accurate estimate in the class of all linear functions of past volatility and past returns.

ACKNOWLEDGMENTS

The author wishes to thank E. Alessio and V. Frappietro for several useful discussions and the Dow Jones Global Indexes for providing the DJIA historical quotes. This work has been supported in part by Federal Program of the Russian Ministry of Industry, Science and Technology No. 40.052.1.1.1112.

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